Research Report on Turaev-Viro Invariants in Finite Rings

Eben Kadile
Under the Supervision of Oleg Viro

August 4, 2019

Abstract

My summer research topic was Turaev-Viro invariants, a class of topological invariants for closed 3-manifolds. These invariants are constructed by fixing some algebraic initial data and considering a simple spine of the 3-manfifold, a special type of cellular collapse obtained after removing a finite number of points from the 3-manifold. I considered new algebraic initial data and discovered an infinite set of simple Turaev-Viro invariants. Here I calculate the invariants for a few simple 3-manifolds and discuss possibilities for further research.

1 Simple Spines of Closed 3-Manifolds

We define a *simple polyhedron* as a topological space where every point's link is one of three possible objects: a circle, a circle with a diameter, or the 1-skeleton of a tetrahedron. These three possibilities are known as the *strata* of the polyhedron: 2-strata, 1-strata, and 0-strata, respectively. We will often refer 2-strata as *faces*, 1-strata as *edges*, and 0-strata as *vertices*.

It is a theorem of [2] that, after a finite number of points are removed, a closed 3-manifold has a cellular collapse which is a simple polyhedron. We call this collapse a *simple spine* of the 3-manifold. One way to obtain a spine is to triangulate the manifold and then replace every tetrahedron with the conified 1-skeleton of a tetrahedron as pictured in Figure 1. This amounts to removing the vertices of the triangulation and then performing a cellular collapse.

It is also a theorem of [2] that the original 3-manifold may always be recovered from its spine by "thickening" the spine. Specifically, if we remove open balls from the manifold instead of points, then this altered manifold is always homeomorphic to a cylinder between its boundary and the spine. A natural question to ask is how the spine may be changed so as to preserve the property that the original 3-manifold may be recovered.

It turns out that this question has a simple answer, found in [3]. There are 3 local "moves" that can generate any transformation of the spine which preserves the original closed 3-manifold. The Bubble move increments/decrements the 2nd Betti number, and corresponds to adding/removing an extra point to the original manifold. The Lune move takes two nearby edges and creates two vertices. The T move takes a structure with 2 vertices and creates 3. See Figure 2.

After a change of basis, the Lune and T moves can be shown to correspond to the Pachner moves acting on the triangulation which was used to obtain the spine.

2 State Sums and Turaev-Viro Invariants

To define Turaev-Viro invariants we must first fix some algebraic initial data.

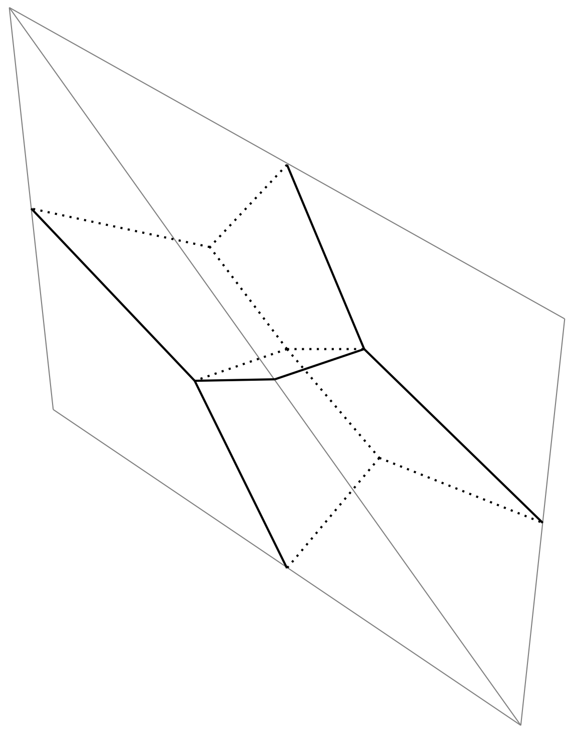

Figure 1: Replacing every tetrahedron in a triangulated 3-manifold with the substructure pictured above will result in a simple spine of the 3-manifold.

Let R be a commutative ring with unity and I be a finite set. Define the *weights* as some function $i \mapsto w_i : I \to R$. Let w be an invertible element of R. Fix a set $adm \in I^3$ which is closed under permutation of the components. We say a triple is *admissible* if it is in adm. We say a 6-tuple (i, j, k, l, m, n) is *admissible* if the triples $(i, j, k), (k, l, m), (m, n, i), (j, l, n)$ are all admissible. We define a *6j-symbol* as a function of admissible 6-tuples:

$$\begin{vmatrix} i & j & k \\ l & m & n \end{vmatrix} \in R$$

Admissible triples should be thought of as ways of "coloring" the faces around an edge (hence the S_3 symmetry) and admissible 6-tuples should be thought of as ways of coloring faces around a vertex. Thus, we must impose the following S_4 symmetry on the 6j-symbol:

$$\begin{vmatrix} i & j & k \\ l & m & n \end{vmatrix} = \begin{vmatrix} j & i & k \\ m & l & n \end{vmatrix} = \begin{vmatrix} i & k & j \\ l & n & m \end{vmatrix} = \begin{vmatrix} i & m & n \\ l & j & k \end{vmatrix} = \begin{vmatrix} l & m & k \\ i & j & n \end{vmatrix} = \begin{vmatrix} l & j & n \\ i & m & k \end{vmatrix}$$

Now suppose we have a simple spine of a closed 3-manifold which was obtained by removing a points. Let the set of faces be $F = \{F_1, ..., F_b\}$. A coloring is a map $\phi : F \to I$. An *admissible* coloring is a coloring such that the triple of colors of the faces around any edge is admissible. Now define the *state sum* as follows:

$$w^{-2a} \sum_{\phi} \Big(\prod_{n=1}^{b} w^2_{\phi(F_n)} \Big) \Big(\prod_{V} \begin{vmatrix} \phi(V_1) & \phi(V_2) & \phi(V_3) \\ \phi(V_4) & \phi(V_5) & \phi(V_6) \end{vmatrix} \Big)$$

Where the sum is taken over all admissible ϕ, the second product is taken over all vertices V of the spine, and $V_1, ..., V_6$ denote the 6 faces around the vertex V.

We can now use the state sum to turn the Bubble move, Lune move, and T move into algebraic equations.

For the Bubble move, for all $i \in I$:

$$w^2 w_i^2 = \sum_{j,k:(i,j,k)\in adm} w_j^2 w_k^2$$

For the Lune move, for all $j_1, j_2, j_3, j_4, j_5, j_6 \in I$ such that

$$(j_1, j_3, j_4), (j_2, j_4, j_5), (j_1, j_3, j_6), (j_2, j_5, j_6) \in adm$$

we must have

$$\sum_j w_j^2 w_{j_4}^2 \begin{vmatrix} j_2 & j_1 & j \\ j_3 & j_5 & j_4 \end{vmatrix} \begin{vmatrix} j_3 & j_1 & j_6 \\ j_2 & j_5 & j \end{vmatrix} = \delta_{j_4, j_6}$$

Where the sum is taken over all j such that all tuples involved are admissible and δ is the Kronecker delta.

For the T move, for all $a, b, c, e, f, j_1, j_2, j_3, j_{23} \in I$ such that $(j_{23}, a, e, j_1, f, b)$ and $(j_3, j_2, j_{23}, b, f, c)$ are admissible we must have

$$\sum_j w_j^2 \begin{vmatrix} j_2 & a & j \\ j_1 & c & b \end{vmatrix} \begin{vmatrix} j_3 & j & e \\ j_1 & f & c \end{vmatrix} \begin{vmatrix} j_3 & j_2 & j_{23} \\ a & e & j \end{vmatrix} = \begin{vmatrix} j_{23} & a & e \\ j_1 & f & b \end{vmatrix} \begin{vmatrix} j_3 & j_2 & j_{23} \\ b & f & c \end{vmatrix}$$

Where, as above, the sum runs over all j such that all tuples involved are admissible.

If the 3 conditions above hold then, by the results discussed in section 1, the state sum must be a topological invariant of the 3-manifold from which the spine was obtained. An invariant of this form is known as a *Turaev-Viro invariant*.

3 My Work

Turaev-Viro invariants are typically found by first taking the ring R to be either $\mathbb{C}$ or some Hopf algebra. My goal was to find a solution to the conditions which is "simple" in an algebraic sense. At first, I was running a search algorithm that I developed. After the program didn't return any substantive results, I wrote down some extra constraints and found a very simple class of solutions.

Let the set of colors $I = \{0, 1\}$. Using the same admissibility conditions as [4] we have 3 possible 6j-symbols (up to symmetry):

$$\begin{vmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{vmatrix}, \begin{vmatrix} 0 & 1 & 1 \\ 0 & 1 & 1 \end{vmatrix}, \begin{vmatrix} 0 & 0 & 0 \\ 1 & 1 & 1 \end{vmatrix}$$

If 1 is the multiplicative identity of our ring then if we set $w_0 = w_1 = 1$ and

$$\begin{vmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{vmatrix} = \begin{vmatrix} 0 & 1 & 1 \\ 0 & 1 & 1 \end{vmatrix} = 1$$

then the Lune and T move amount to the simple constraint:

$$\begin{vmatrix} 0 & 0 & 0 \\ 1 & 1 & 1 \end{vmatrix}^2 = 1$$

If we were to take the ring R to be the integers, solutions similar to setting the final 6j-symbol equal to ± 1 were studied in [4] and are related to the Betti numbers of the manifold. This makes the case of $R = \mathbb{Z}$ seem rather trivial.

However, if n is an integer greater than 2, then the ring $\mathbb{Z}/2^n\mathbb{Z}$ has 4 square roots of unity: ± 1 and $2^{n-1} \pm 1$. We suspect that the case of setting the last 6j-symbol equal to $2^{n-1} \pm 1$ might yield interesting new information about the manifold.

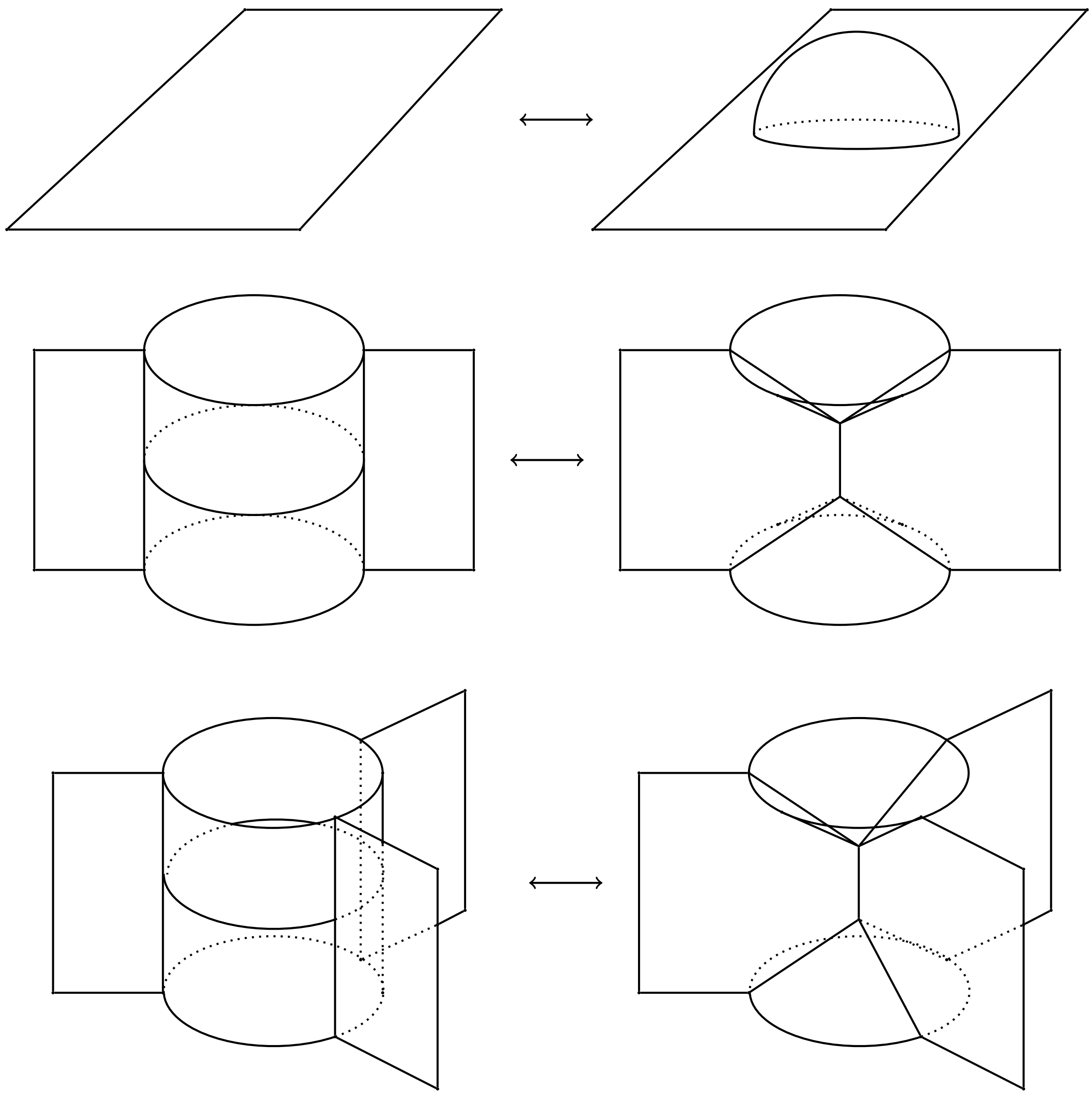

Figure 2: The Bubble move, Lune move, and T move, respectively.

4 Computation of the Invariants for Some Simple 3-Manifolds

The 3-manifolds S^3, $\mathbb{R}P^3$, $S^2 \times S^1$, and $L(3,1)$ all admit simple spines without vertices. Thus, their Turaev-Viro invariants depend only on the weights, and were computed in [4]. The weights for my invariant are trivial, so I will not bother computing it for these spaces. All computations are done using the figures and tables of [1].

The lens spaces $L(4,1)$ and $L(5,2)$ admit simple spines with at least one vertex. There are two admissible colorings of the spine for $L(4,1)$, they correspond to the 6j-symbols which were set equal to 1. Thus, for any $n > 2$, the value of the invariant for $L(4,1)$ in $\mathbb{Z}/2^n\mathbb{Z}$ is 2. There is only one admissible coloring of the spine for $L(5,2)$, it is equivalent to simply labeling every face with the value 0. This means that the value for this space is 1. As trivial as it may be, this shows us that Turaev-Viro invariants can easily distinguish between Lens spaces.

The space S^3/Q_8, where Q_8 is the quaternion group, has a simple spine with two vertices. There are four admissible colorings, and all the 6j-symbols involved are 1, so the value of invariant is 4. Again, we're able to distinguish between quotients of S^3, but none of the colorings so far involve the 6j-symbol whose value is some square root of unity. The manifold S

The manifold $S^3/(Q_8 \times Z_3)$ has a simple spine with four vertices. There are two admissible colorings, one which colors every face with 0 and another which has two 6j-symbols whose values are the square root of unity. Thus, the value of the invariant for this space is 2.

$S^3/(Q_8 \times Z_5)$ has a simple spine with five vertices. There are four admissible colorings. For this spine, there are also colorings where the 6j-symbols take the value of the selected square root of unity. However, there is always two or zero of them in each coloring. Therefore, the value of the invariant for this space is 4.

If we take K to be the Klein bottle then we can construct a spine for $K \times S^1$. If we remove one point from $K \times S^1$ then we can colllapse all except one K fiber to a circle with a diameter. That is, if D is the circle with diameter, we can obtain a spine for $K \times S^1$ by attaching $D \times [0,1]$ to K by gluing $D \times \{0\}$ and $D \times \{1\}$ to K such that they have exactly one intersection point. There are two admissible colorings of this spine, one assigns 0 to every face and the other has four 6j-symbols which take the value of our selected square root of unity. Therefore, the value of the invariant for $K \times S^1$ is 2.

5 Future Work

There are a few directions that this research could be taken in.

In Section 4, we found the value of the invariants for manifolds whose spines only have colorings where the 6j-symbol which is a square root of unity occurs an even number of times. It would be interesting to find manifolds where it occurs an odd number of times and determine what characterizes these manifolds. If there are no such manifolds, it would be interesting to see how this fact is proven.

Another direction is to try to develop a (co)homology theory from which the invariants can be re-obtained in a similar way to re-obtaining the Euler characteristic from singular homology. Such a homology theory would give us more information than the invariants by themselves.

References

[1] Matveev, Sergei, *Algorithmic Topology and Classification of 3-Manifolds.* 2nd Edition. Algorithms and Computation in Mathematics, Vol 9. Springer, 2007.

[2] Matveev, Sergei, *Special spines of piecewise linear manifolds.* Mat. sbornik 92 (134) (1973), 282293.

[3] Piergallini, R, *Standard moves for standard polyhedra and spines.* III Convegno Nazionale di Topologia Trieste, 912 Giugno 1986 (1988), 391414.

[4] Turaev, V.G. and O.Y. Viro, *State Sum Invariants of 3-Manifolds and Quantum 6j-Symbols.* Topology, Vol. 31. Pergamon Press Ltd, 1992.